Directed motion in a periodic potential of a quantum system coupled to a heat bath driven by a colored noise

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A system-reservoir nonlinear coupling model is proposed for a quantum system when the associated bath is not in thermal equilibrium but is modulated by an external colored noise, to present a microscopic approach to quantum state-dependent diffusion and multiplicative noise in terms of a quantum Langevin description. Consequently, the Fokker-Planck equation in position space, valid in the overdamped limit, for multiplicative colored noise is constructed to explore the possibility of observing a quantum current and dependence of the current on various parameters of external noise is examined.

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I. INTRODUCTION

Thermal diffusion is an actively pursued area of research. In a periodic potential it has an important role when studying Josephson's junctions [1], oscillators with noisy limit cycles [2], diffusion in crystal surfaces [3], and many others. In contemporary research it has been followed with a lot of interest while studying the transport properties of Brownian particles moving in a periodic potential [4] with special stress on giant diffusion and coherent transport [5].

The motivation for all these studies partly lies in an attempt to understand how protein motors move in biological systems [6]. To understand such transport phenomena, various models have been proposed such as the vibrational ratchet [7], rocking ratchet [8], flashing ratchet [5], diffusion ratchet [9,10], correlation ratchet [11], and others. These models have large scale applications in nanoscopic systems and biology [12] due to their effectiveness in understanding experimental observations on biochemical motors active in muscle contraction [13], directed transport in photovoltaic and photoreflective materials [14], and others. The potential in all these models is taken to be asymmetric in space. A unidirectional current can also be obtained from a spatially symmetric potential. In such nonequilibrium systems, time asymmetric random forces or space-dependent diffusion is required [15]. The space-dependent friction coefficient or space dependent temperature may lead to the emergence of the space-dependent diffusion coefficient [16–18]. In superlattice structures, semiconductors, or motion in porous media, frictional inhomogenities are common. Space-dependent friction is experienced by particles moving close to a surface [19,20].

In 1987, Bütikker [18] had shown that a classical particle experiences a net drift force resulting in the generation of current if the particle is in a symmetric sinusoidal potential field in the presence of sinusoidally modulated space-dependent diffusion with the same periodicity. Bütikker had shown this in the case of space-dependent friction in the overdamped limit where a directional mass flow may be obtained. The origin of this current is basically the phase difference between the symmetric periodic potential and space-dependent diffusion. The current vanishes for phase difference of zero and multiples of π. van Kampen [21] had come to similar conclusions in a latter work for systems in overdamped condition with space-dependent temperature. The problem of Langevin equation with multiplicative noise and state-dependent dissipation for a thermodynamical closed system has been well studied. The classical quantum-mechanical system reservoir linear coupling model for microscopic description of additive noise and linear dissipation which are related by the fluctuation dissipation relation (FDR) is well known over many decades in several fields [22,23], the nature of nonlinear coupling and its consequences have been explored with renewed interest only recently. For example, the nonlinear coupling approach has been extensively used by Tanimura and co-workers [24] in explaining elastic and inelastic relaxation mechanisms along with their effects on vibrational and Raman spectroscopy. Without using the rotating wave approximation for the system-bath coupling, recently they have developed [25] a quantum dissipative equation with Gaussian-Markovian noise that has applicability to low-temperature systems strongly coupled to a harmonic bath. But in all such cases, the corresponding Langevin equation has been considered for a thermodynamical closed system. A Langevin equation with state-dependent dissipation and multiplicative colored noise processes for an open system has drawn little attention apart from a few exceptions [26,27].

In the classical regime, the transport of macroscopic objects such as Brownian particles is well elaborated in literature [28], special interest has been devoted to transport in ratchet systems (also termed Brownian motor systems) [29]. In contrast, the quantum properties of directed transport are only partially elaborated in such motor systems [30,31]. Challenges arise in the quantum region because the transport can strongly depend on the mutual interplay of pure quantum

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effects such as tunneling and particle wave interference with dissipation process, nonequilibrium fluctuation, and external driving [32]. An important concept introduced recently in classical and quantum Hamiltonian transport is that for a spatially periodic system in which, without a biased force, a directed current of particles can be established. Ratchet models were originally proposed as a mechanism for some kinds of biological motors and as nanoscale devices for several applications [29]. In these and other contexts [33], the directed current is due to a spatial or temporal asymmetry combined with noise and dissipation. In a classical Hamiltonian system, dissipation is absent and noise is replaced by deterministic chaos. Here, a directed current of particles in the chaotic sea may arise under asymmetry conditions for a mixed phase space [34]. The corresponding quantized system may exhibit a significant ratchet behavior, even in a fully chaotic region [35]. Such a behavior, which occurs in a variant of the kicked rotor and can be related to the underlying classical dynamics, was observed recently in experiment using ultracold atoms [36]. Very recently, an experimental realization of quantum ratchets associated with quantum resonance of the kicked particle for arbitrary values of the quasimomentum has been reported [37]. However, the theoretical study of the phenomena of quantum ratchets remains wide open. In an earlier work, Landauer [38,39] explored the problem of characterizing nonequilibrium states in the transition kinetics between two locally stable states in bistable systems. van Kampen's study [21] is a reevaluation of this earlier work.

In the present work, we address the problem of quantum Langevin equation with multiplicative noise and state-dependent diffusion for a thermodynamically open system to explore the nature of nonlinear coupling and modulation of heat bath and its consequences, specifically the possibility of observing the directed transport in a periodic potential as a consequence of state-dependent dissipation. We consider a system-reservoir model where the associated bath is not in thermal equilibrium but is modulated by external colored noise with an exponentially decaying correlation function and the system is nonlinearly coupled with the heat bath, thereby resulting in a nonlinear multiplicative quantum Langevin equation with state-dependent dissipation. When the reservoir is modulated by an external noise, it is likely that it induces fluctuations in the polarization of the reservoir [26]. Due to the presence of external noise, one may expect that the nonequilibrium situation created by modulating the bath may induce an asymmetry in the effective potential [26,40]. A number of different situations depicting the modulation of the heat bath may be physically relevant. For example, we may consider a simple unimolecular conversion $X \to Y$, say, an isomerization reaction, carried out in a photochemically active solvent. The growth in living polymerization [41] is another such example. Since the fluctuations in the light intensity result in fluctuation in the polarization of the solvent molecules, the effective reaction field around the reactants gets modified.

It is pertinent here to mention the fact that obtaining the quantum reaction rate and studying the quantum transport phenomena in macroscopic systems is a challenging task, and many authors resort to a classical or semiclassical approach as a tool for studying the dynamics. However, the Monte Carlo methods which are essential for obtaining numerically exact quantum rates have thus far largely eluded quantum dynamics. The averaging over a large number of oscillatory terms, even with today's computers, does not converge. The impressive state of the art computations on dissipative systems [42] remain limited and are not readily generalized to "realistic" systems [43]. Very recently, formulation of a quantum Langevin equation based on C-number approach [44] has been proposed where the authors have used a coherent state representation of the noise operator and a canonical thermal Wigner distribution of the bath oscillators. This formalism and its different variants [44,45] have been applied successfully to explain several aspects of reaction rate theory in condensed phases within the quantum-mechanical context. Prompted by its success, we explore in this paper a physically motivated formalism in the context of quantum-mechanical Langevin equation with state-dependent dissipation and multiplicative noise to study the transport of a quantum system in a periodic and symmetric potential.

The organization of the paper is as follows. Following the recently developed methodology by Ray and co-workers [44], starting from a microscopic Hamiltonian picture of a quantum system nonlinearly coupled with a harmonic bath which is modulated by an external noise, we derive the C-number analog of the quantum Langevin equation for the system mode in Sec. II. In Sec. III, the Fokker-Planck description of the Langevin equation with state-dependent dissipation and multiplicative colored noise is provided followed by a Smoluchowski description of the process. In Sec. IV, as an application of our development, we derive the net quantum current in a sinusoidal symmetric potential and the various characteristics of the current is explored. The paper is concluded in Sec. V.

II. MODEL AND QUANTUM LANGEVIN EQUATION

Our model consists of a particle of unit mass nonlinearly coupled to a heat bath consisting of N-harmonic oscillators driven by an external noise. The total Hamiltonian for such a composite system can be written as

$$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{q}) + \sum_{j=1}^{N} \left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\omega_j^2[\hat{x}_j - c_j f(\hat{q})]^2 \right] + H_{\rm int}, \quad (1)$$

where $\hat{q}$ and $\hat{p}$ are the coordinate and momentum operators of the system, respectively, $V(\hat{q})$ is the potential energy operator and $\{\hat{x}_j, \hat{p}_j\}$ are the set of coordinate and momentum operators for the jth bath oscillators having characteristic frequency ω_j. The masses of the bath oscillators are chosen to be unity. The system particle is nonlinearly coupled to the bath oscillators through the general coupling term $c_j f(\hat{q})$, where c_j is the coupling constant. $H_{\rm int}$ is the interaction term between the heat bath and the external classical noise $\epsilon(t)$, with the following form:

$$H_{\rm int} = \sum_{j=1}^{N} \kappa_j \hat{x}_j \epsilon(t). \tag{2}$$

In Eq. (2), κ_j denotes the strength of the interaction. We consider $\epsilon(t)$ to be a stationary Gaussian noise process with zero mean and arbitrary correlation function

$$\langle \epsilon(t) \rangle = 0, \quad \langle \epsilon(t)\epsilon(t') \rangle = 2D\psi(t-t'), \tag{3}$$

where D is the external noise strength and $\psi(t)$ is the external noise memory kernel which is assumed to be a decaying function of its argument. The coordinate and momentum operator satisfy the usual commutation relations

$$[\hat{q},\hat{p}] = i\hbar \quad \text{and} \quad [\hat{x}_j,\hat{p}_k] = i\hbar\,\delta_{jk}. \tag{4}$$

The physical situation we address here is the following. Initially (i.e., at $t=0$), the system and the reservoir is in thermal equilibrium at temperature T. At $t=0_+$, the external noise agency is switched on and the bath is modulated by the external noise. We follow the dynamics of the system for subsequent times.

We now use Eq. (1) to obtain the following dynamical equations for the position and momentum operators:

$$\dot{\hat{q}} = \hat{p}, \tag{5}$$

$$\dot{\hat{p}} = -V'(\hat{q}) + f'(\hat{q}) \sum_{j=1}^{N} c_j \omega_j^2 [\hat{x}_j - c_j f(\hat{q})], \tag{6}$$

where the overdot indicates derivative with respect to time and the prime refers to derivative with respect to $\hat{q}$. Similarly, we have the dynamical equation of motion for the bath oscillators $(j=1,2,\ldots,N)$

$$\dot{\hat{x}}_j = \hat{p}_j, \tag{7}$$

$$\dot{\hat{p}}_j = -\omega_j^2 [\hat{x}_j - c_j f(\hat{q})] - \kappa_j \epsilon(t) \hat{I}, \tag{8}$$

where $\hat{I}$ is the unit operator. To eliminate the bath degrees of freedom from the equation of motion of the system, we first obtain a solution for the position operator $\hat{x}_j$ by formally solving the above equations and then make use of the solution in Eq. (6) followed by an integration by parts. This yields the generalized operator Langevin equation for the system particle:

$$\dot{\hat{q}} = \hat{p}, \tag{9}$$

$$\dot{\hat{p}} = -V'(\hat{q}) - f'(\hat{q}) \int_0^t \gamma(t-t') f'[\hat{q}(t')] \hat{p}(t') dt' + f'[\hat{q}(t)] \hat{\eta}(t) + f'[\hat{q}(t)] \pi(t), \tag{10}$$

where the noise operator $\hat{\eta}(t)$ and the memory kernel $\gamma(t)$ are given by

$$\hat{\eta}(t) = \sum_{j=1}^{N} c_j \omega_j^2 \left\{ \{\hat{x}_j(0) - c_j f[\hat{q}(0)]\} \cos(\omega_j t) + \frac{\hat{p}_j(0)}{\omega_j} \sin(\omega_j t) \right\} \tag{11}$$

and

$$\gamma(t) = \sum_{j=1}^{N} c_j^2 \omega_j^2 \cos(\omega_j t). \tag{12}$$

In Eq. (10), $\pi(t)$ is the fluctuating force generated due to external stochastic driving $\epsilon(t)$ and is given by

$$\pi(t) = -\int_0^t dt' \varphi(t-t') \epsilon(t'), \tag{13}$$

where

$$\varphi(t) = \sum_{j=1}^{N} c_j \omega_j \kappa_j \sin(\omega_j t). \tag{14}$$

The form of Eq. (10) indicates that the system is driven by two fluctuating forces $\hat{\eta}(t)$ and $\pi(t)$ which are multiplicative in nature due the presence of a function of system variable $f'(\hat{q})$. The statistical properties of $\hat{\eta}(t)$ can be derived by using suitable canonical thermal distribution of bath coordinates and momentum operators at $t=0$. We assume that the initial distribution is one in which the bath is equilibrated at $t=0$ in the presence of the system but in the absence of external noise agency $\epsilon(t)$ so that

$$\langle \hat{\eta}(t) \rangle_{\rm QS} = 0, \tag{15}$$

$$\frac{1}{2}\langle \hat{\eta}(t)\hat{\eta}(t') + \hat{\eta}(t')\hat{\eta}(t) \rangle_{\rm rmQS} = \sum_j \frac{1}{2} c_j^2 \omega_j^2 \hbar\omega_j \coth(\hbar\omega/2k_B T) \cos \omega_j (t-t'). \tag{16}$$

Here, $\langle \cdots \rangle_{\rm QS}$ implies quantum statistical average over the bath degrees of freedom and is defined as

$$\langle \hat{A} \rangle_{\rm QS} = \frac{\mathrm{Tr}[\hat{A} \exp(-\hat{H}_B/k_B T)]}{\mathrm{Tr}[\exp(-\hat{H}_B/k_B T)]} \tag{17}$$

for any bath operator $\hat{A}(\hat{x}_j,\hat{p}_j)$, where

$$H_B = \sum_{j=1}^{N} \left[\frac{\hat{p}_j^2}{2} + \frac{1}{2}\omega_j^2 [\hat{x}_j - c_j f(\hat{q}_j)]^2 \right] \text{ at } t=0.$$

By trace we mean the usual quantum statistical average. Equation (16) is the fluctuation-dissipation relation expressed in terms of noise operators appropriately ordered in quantum-mechanical sense.

Following Ray *et al.* [44,45], to construct a C-number quantum Langevin equation, we carry out a quantum-mechanical average of Eq. (10), where the quantum-mechanical average $\langle \cdots \rangle_Q$ is taken over the initial product separable quantum states of the system and the bath oscillators at $t=0$, $|\phi\rangle\{|\alpha_j\rangle\}$; $j=1,2,\cdots,N$. Here, $|\phi\rangle$ denotes any arbitrary initial state of the system and $|\alpha_j\rangle$ corresponds to the initial coherent state of the jth bath oscillator. We then obtain

$$\langle \hat{q} \rangle_Q = \langle \hat{p} \rangle_Q, \tag{18}$$

$$\langle \hat{p} \rangle_Q = -\langle V'(\hat{q}) \rangle_Q - \left\langle f'[\hat{q}(t)] \int_0^t dt' \gamma(t-t') f'[\hat{q}(t')] \hat{p}(t') \right\rangle_Q + \langle f'[\hat{q}(t')] \hat{\eta}(t) \rangle_Q + \langle f'[\hat{q}(t')] \rangle_Q \pi(t). \tag{19}$$

Since $\hat{\eta}(t)$ contains operators at time $t=0$, Eq. (19) can be rewritten as

$$\langle \hat{p} \rangle_Q = -\langle V'(\hat{q}) \rangle_Q - \left\langle f'[\hat{q}(t)] \int_0^t dt' \gamma(t-t') f'[\hat{q}(t')] \hat{p}(t') \right\rangle_Q + \langle f'[\hat{q}(t')] \rangle_Q \langle \hat{\eta}(t) \rangle_Q + \langle f'[\hat{q}(t')] \rangle_Q \pi(t). \tag{20}$$

$\langle \hat{\eta}(t) \rangle_Q$ is now a classical-like noise term, which, in general, is a nonzero number because of the quantum-mechanical averaging and is given by

$$\langle \hat{\eta}(t) \rangle_Q = \sum_{j=1}^{N} \left[c_j \omega_j^2 \left\{ [\langle \hat{x}_j(0) \rangle_Q - c_j \langle f[\hat{q}(0)] \rangle_Q] \cos(\omega_j t) + \frac{\langle \hat{p}_j(0) \rangle_Q}{\omega_j} \sin(\omega_j t) \right\} \right]. \tag{21}$$

To realize $\langle \hat{\eta}(t) \rangle_Q$ as an effective C-number noise term, Ray *et al.* [44,45] proposed that the momentum $\langle \hat{p}_j(0) \rangle_Q$ and the shifted coordinate $(\langle \hat{x}_j(0) \rangle_Q - c_j \langle f[\hat{q}(0)] \rangle_Q)$ of the bath oscillators be distributed according to the canonical distribution of Gaussian form

$$\mathcal{P}_j = \mathcal{N} \exp\left\{ - \frac{[\langle \hat{p}_j(0) \rangle_Q^2 + \omega_j^2 \{\langle \hat{x}_j(0) \rangle_Q - c_j \langle f[\hat{q}(0)] \rangle_Q\}^2]}{2\hbar \omega_j \left[\bar{n}_j(\omega_j) + \frac{1}{2} \right]} \right\} \tag{22}$$

so that for any quantum-mechanical mean value of the operator $\langle \hat{A} \rangle_Q$ which is a function of bath variables, its statistical average $\langle \cdots \rangle_S$ is

$$\langle \langle \hat{A} \rangle_Q \rangle_S = \int [\langle \hat{A} \rangle_Q \mathcal{P}_j d(\omega_j^2 \{\langle \hat{x}_j(0) \rangle_Q - c_j \langle f(\hat{q}(0)) \rangle_Q\})] d\langle \hat{p}_j(0) \rangle_Q \tag{23}$$

In Eq. (22), $\bar{n}_j(\omega_j)$ is the average thermal photon number of the jth bath oscillator at temperature T and is given by

$$\bar{n}_j(\omega_j) = \frac{1}{\exp(\hbar \omega_j / k_B T - 1)}. \tag{24}$$

The distribution $\mathcal{P}_j$ given by Eq. (22) and the statistical average as defined in Eq. (23) indicate that the C-number noise $\langle \hat{\eta}(t) \rangle_Q$ must satisfy

$$\langle \langle \hat{\eta}(t) \rangle_Q \rangle_S = 0 \tag{25}$$

and

$$\langle \langle \hat{\eta}(t) \hat{\eta}(t') \rangle_Q \rangle_S = \frac{1}{2} \sum_{j=1}^{N} c_j^2 \omega_j^2 \hbar \omega_j \coth(\hbar \omega_j / 2 k_B T) \cos \omega_j (t-t'). \tag{26}$$

Now we must impose some conditions on the coupling coefficients c_j and κ_j, on the bath frequencies ω_j and on the number N of the bath oscillators that will ensure that $\gamma(t)$ is indeed dissipative. A sufficient condition for the $\gamma(t)$ to be dissipative is that it be positive definite and decrease monotonically with time. These conditions are achieved if $N \to \infty$ and if $c_j \omega_j^2$ and ω_j are sufficiently smooth functions of j [46]. As $N \to \infty$, one replaces the sum by an integral over ω weighted by a density of state $\mathcal{D}(\omega)$. Thus to obtain a finite result in the continuum limit, the coupling function $c_j = c(\omega)$ and $\kappa_j = \kappa(\omega)$ are chosen as $c(\omega) = \frac{c_0}{\omega \sqrt{\tau_c}}$ and $\kappa(\omega_j) = \kappa_0 \omega \sqrt{\tau_c}$. Consequently, $\gamma(t)$ and $\varphi(t)$ reduce to the following form:

$$\gamma(t) = \frac{c_0^2}{\tau_c} \int_0^\infty d\omega \mathcal{D}(\omega) \cos(\omega t) \tag{27}$$

and

$$\varphi(t) = c_0 \kappa_0 \int_0^\infty d\omega \mathcal{D}(\omega) \omega \sin(\omega t), \tag{28}$$

where c_0 and κ_0 are constants and $\frac{1}{\tau_c}$ is the cutoff frequency of the bath oscillators. τ_c may be regarded as the correlation time of the bath and $\mathcal{D}(\omega)$ is the density of modes of the heat bath which is assumed to be Lorentzian:

$$\mathcal{D}(\omega) = \frac{2}{\pi \tau_c (\omega^2 + \tau_c^{-2})}. \tag{29}$$

With these forms of $\mathcal{D}(\omega)$, $c(\omega)$, and $\kappa(\omega)$, $\gamma(\omega)$ and $\varphi(\omega)$ take the following forms:

$$\gamma(t) = \frac{c_0^2}{\tau_c} \exp(-t/\tau_c) = \frac{\Gamma}{\tau_c} \exp(-t/\tau_c) \tag{30}$$

and

$$\varphi(t) = \frac{c_0 \kappa_0}{\tau_c} \exp(-t/\tau_c) \tag{31}$$

with $\Gamma = c_0^2$. For $\tau_c \to 0$, Eqs. (30) and (31) give $\gamma(t) = 2\Gamma \delta(t)$ and $\varphi(t) = 2 c_0 \kappa_0 \delta(t)$. At this juncture we consider that the external noise $\epsilon(t)$ is an Ornstein-Uhlenbeck process [23] with $\langle \epsilon(t) \epsilon(t') \rangle = \frac{D}{\tau} \exp(-|t-t'|/\tau)$, then from Eq. (13) it is easy to obtain that the dressed noise $\pi(t)$ has the correlation function

$$\langle \pi(t) \pi(t') \rangle = \frac{D \Gamma \kappa_0^2}{\tau} \exp(-|t-t'|/\tau), \tag{32}$$

where D is the strength of the noise $\epsilon(t)$ and τ its correlation time. On the other hand, the noise correlation function (26) becomes

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = \frac{\Gamma}{2\tau_c}\int_0^\infty d\omega\hbar\omega\coth(\hbar\omega/2k_BT) \times\cos\omega(t-t')\mathcal{D}(\omega). \quad (33)$$

Equation (33) is an exact expression for quantum statistical average of two time correlation functions of $\hat{\eta}(t)$. We now make the following approximation. As $\hbar\omega\coth(\hbar\omega/2k_BT)$ is a much more smooth function of ω, at least for not too low temperatures, the integral can be approximated as

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S \simeq \frac{\Gamma}{2\tau_c}\hbar\omega_0\coth(\hbar\omega_0/2k_BT) \times\int_0^\infty d\omega\cos\omega(t-t')\mathcal{D}(\omega), \quad (34)$$

where ω_0 is the average frequency of the bath. This approximation is well known and frequently used in quantum optics for the weak coupling scheme [22]. With this approximation, the underlying quantum noise process reduces to Markovian noise [47]. Defining

$$D_0 = \frac{\Gamma}{2}\hbar\omega_0\left(\bar{n}(\omega_0)+\frac{1}{2}\right), \quad (35)$$

the above expression reduces to

$$\langle\langle\hat{\eta}(t)\hat{\eta}(t')\rangle_Q\rangle_S = 2D_0\delta(t-t') \text{ for } \tau_c\to 0, \quad (36)$$

where $\mathcal{D}$ is given by Eq. (29). Here, it is pertinent to note that our abovementioned assumption is not valid at very low temperatures. In this sense, our development cannot be claimed to be fully quantum, rather a quasiclassical one. Nevertheless, the ansatz define by Eq. (22), which is the canonical thermal Wigner distribution function for a shifted harmonic oscillator [48] and always remains a positive definite function, contains the quantum information of the bath. A special advantage of using this distribution function is that it remains valid as a pure state nonsingular distribution function even at $T=0$. Thus, from the very mode of our development it is clear that apart from the calculation of two time correlation functions, Eq. (36), the rest of our treatment is truly quantum mechanical.

Writing $q=\langle\hat{q}\rangle_Q$ and $p=\langle\hat{p}\rangle_Q$ for brevity, we can now rewrite Eqs. (18) and (20) as

$$\dot{q} = p, \quad (37)$$

$$\dot{p} = -\langle V'(\hat{q})\rangle_Q - \Gamma\langle[f'(\hat{q})]^2\hat{p}\rangle_Q + \langle f'(\hat{q})\rangle_Q[\eta(t)+\pi(t)], \quad (38)$$

where $\eta(t)=\langle\hat{\eta}(t)\rangle_Q$ and is a classical-like noise term. In writing Eq. (38) we have made use of the fact that the correlation time of the reservoir is very short, i.e., $\tau_c\to 0$.

We now define an effective noise $\xi(t)=\eta(t)+\pi(t)$. The effective noise $\xi(t)$ will have an intensity D_R and correlation time τ_R given by

$$D_R = \int_0^\infty\langle\xi(t)\xi(0)\rangle dt,$$

$$\tau_R = \frac{1}{D_R}\int_0^\infty\langle\xi(t)\xi(0)\rangle dt, \quad (39)$$

where the averaging is taken over each realization of $\eta(t)$ and $\pi(t)$ independently. Following the above definitions we obtain

$$D_R = \Gamma(D_0 + D\kappa_0^2) \text{ and } \tau_R = \frac{\Gamma D\kappa_0^2}{D_R}\tau = \frac{D\kappa_0^2\tau}{D_0 + D\kappa_0^2}. \quad (40)$$

In terms of the effective noise $\xi(t)$, Eqs. (37) and (38) become

$$\dot{q} = p,$$

$$\dot{p} = -\langle V'(\hat{q})\rangle_Q - \Gamma\langle[f'(\hat{q})]^2\hat{p}\rangle_Q + \langle f'(\hat{q})\rangle_Q\xi(t), \quad (41)$$

where $\xi(t)$ is a C-number Gaussian Ornstein-Uhlenbeck–type noise [23] so that

$$\langle\xi(t)\rangle = 0,$$

$$\langle\xi(t)\xi(t')\rangle = \frac{D_R}{\tau_R}\exp(-|t-t'|/\tau_R), \quad (42)$$

where D_R and τ_R are given by Eq. (40).

We now add $V'(q)$, $\Gamma[f'(q)]^2p$, and $f'(q)\xi(t)$ on both sides of Eq. (41) and rearrange it to obtain

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) + Q_V - \Gamma[f'(q)]^2p + Q_1 + f'(q)\xi(t) + Q_2, \quad (43)$$

where

$$Q_V = V'(q) - \langle V'(\hat{q})\rangle_Q,$$

$$Q_1 = \Gamma[f'(q)]^2p - \langle[f'(\hat{q})]^2\hat{p}\rangle_Q,$$

$$Q_2 = \xi(t)[\langle f'(\hat{q})\rangle_Q - f'(\hat{q})]. \quad (44)$$

Referring to the quantum nature of the system in the Heisenberg picture, we now write the system operator $\hat{q}$ and $\hat{p}$ as

$$\hat{q} = q + \delta\hat{q},$$

$$\hat{p} = p + \delta\hat{p}, \quad (45)$$

where $q(=\langle\hat{q}\rangle_Q)$ and $p(=\langle\hat{p}\rangle_Q)$ are the quantum-mechanical mean values and $\delta\hat{q}$ and $\delta\hat{p}$ are the operators signifying quantum fluctuations around the respective mean value. By construction, $\langle\delta\hat{q}\rangle_Q=\langle\delta\hat{p}\rangle_Q=0$ and they also follow the usual commutation relation $[\delta\hat{p},\delta\hat{p}]=i\hbar$. Using Eq. (45) in $V'(\hat{q})$, $[f'(\hat{q})]^2\hat{p}$, and in $f'(\hat{q})$, a Taylor series expansion in $\delta\hat{q}$ around q, Q_V, Q_1 and Q_2 can be obtained as

$$Q_V = -\sum_{n\geq 2}\frac{1}{n!}V^{n+1}(q)\langle\delta\hat{q}^n\rangle_Q, \quad (46)$$

$$Q_1 = -\Gamma[2pf'(\hat{q})Q_f + pQ_3 + 2f'(\hat{q})Q_4 + Q_5], \quad (47)$$

$$Q_2 = \xi(t)Q_f, \tag{48}$$

where

$$Q_f = \sum_{n\geq 2} \frac{1}{n!} f^{n+1}(q)\langle \delta\hat{q}^n \rangle_Q,$$

$$Q_3 = \sum_{m\geq 2}\sum_{n\geq 2} \frac{1}{m!}\frac{1}{n!} f^{m+1}(q) f^{n+1}(q)\langle \delta\hat{q}^m \delta\hat{q}^n \rangle_Q,$$

$$Q_4 = \sum_{n\geq 2} \frac{1}{n!} f^{n+1}(q)\langle \delta\hat{q}^n \delta\hat{p} \rangle_Q,$$

$$Q_5 = \sum_{m\geq 2}\sum_{n\geq 2} \frac{1}{m!}\frac{1}{n!} f^{m+1}(q) f^{n+1}(q)\langle \delta\hat{q}^m \delta\hat{q}^n \delta\hat{p} \rangle_Q. \tag{49}$$

From the above expression it is evident that Q_V represents quantum correction due to the nonlinearity of the system potential, Q_1 and Q_2 reflect the quantum corrections due to nonlinearity of the system bath coupling function. Using Eqs. (46)–(48), we get the dynamical equations for the system variable from Eq. (43) as

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) + Q_V - \Gamma[f'(q)]^2 p - 2\Gamma p f'(q) Q_f - \Gamma p Q_3 - 2\Gamma p f'(q) Q_4 - \Gamma Q_5 + f'(q)\xi(t) + Q_f \xi(t). \tag{50}$$

It is well documented in the literature [49] that when the fluctuation is state dependent or equivalently when the noise is multiplicative with respect to the system variable, which is a manifestation of the nonlinear nature of the system-bath coupling function, the conventional adiabatic elimination of the fast variable in the overdamped limit does not provide correct results. To obtain a correct equilibrium distribution, Sancho *et al.* [50] had proposed an alternative approach in the case of multiplicative noise system. By carrying out a systematic expansion of the relevant variables in powers of Γ^{-1} and neglecting terms smaller than $O(\Gamma^{-1})$, they obtained the dynamical equation of motion for position coordinate. We follow the same procedure in our context. In this limit, the transient correction terms Q_4 and Q_5 do not affect the dynamics of the position which varies in a much more slower time scale in the overdamped limit [44]. So the equations governing the dynamics of the system variables are

$$\dot{q} = p,$$

$$\dot{p} = -V'(q) + Q_V - \Gamma h(q) p + g(q)\xi(t), \tag{51}$$

where

$$h(q) = [f'(q)]^2 p + 2f'(q)Q_f + Q_3, \tag{52}$$

$$g(q) = f'(q) + Q_f. \tag{53}$$

The function $g(q)$ arises due to nonlinearity of the system-bath coupling function $f(q)$, where Q_f is the quantum correction to the classical contribution $f'(q)$. For a linear coupling function $g(q)$ reduces to a constant.

We can now easily identify Eq. (51) as the C-number analog of the quantum Langevin equation, where $\Gamma h(q)$ is the state-dependent damping and $\xi(t)$, the effective noise which arises due to mutual effect of system bath interaction and bath modulation. The statistical properties of $\xi(t)$ were given earlier [see Eqs. (39) and (40)].

III. THE FOKKER-PLANCK DESCRIPTION

To obtain the Fokker-Planck equation corresponding to Eq. (51) we rewrite it as

$$\dot{u}_1 = G_1[u_1, u_2, t; \xi(t)],$$

$$\dot{u}_2 = G_2[u_1, u_2, t; \xi(t)], \tag{54}$$

where we have used the following abbreviation $u_1 = q$, $u_2 = p$ and $G_1 = p$, $G_2 = -U'(q) - \Gamma h(q) p + g(q)\xi(t)$ with $U'(q) = V' - Q_V$.

The vector u with components u_1 and u_2 thus represents a point in a two-dimensional "phase space" and Eq. (54) determines the velocity at each point in this phase space. The conservation of the points now asserts the following linear equation of motion for density $\rho(u,t)$ in phase space:

$$\frac{\partial}{\partial t}\rho(u,t) = -\sum_{n=1}^{2} \frac{\partial}{\partial u_n} G_n[u_1, u_2, t; \xi(t)]$$

or, more compactly,

$$\frac{\partial \rho(u,t)}{\partial t} = -\nabla \cdot G\rho. \tag{55}$$

Our next task is to find out a differential equation whose average solution is given by $\langle \rho \rangle$ where the stochastic average has to be performed over two noise processes $\eta(t)$ and $\xi(t)$. $\nabla \cdot G$ can be partitioned into two parts: a constant part $\nabla \cdot G_0$ and a fluctuating part $\nabla \cdot G_1(t)$, containing these noises. Thus, we write

$$\nabla \cdot G[u_1, u_2, \eta(t), \epsilon(t)] = \nabla \cdot G_0(u_1, u_2) + \alpha \nabla \cdot G_1[u_1, u_2, t; \eta(t), \epsilon(t)], \tag{56}$$

where α is a parameter externally introduced to keep track of the order of the perturbation expansion (we put $\alpha = 1$ at the end of the calculation). Equation (55) thus takes the form

$$\dot{\rho}(u_1, u_2, t) = (A_0 + \alpha A_1)\rho(u_1, u_2, t), \tag{57}$$

where $A_0 = -\nabla \cdot G_0$ and $A_1 = -\nabla \cdot G_1$. The symbol ∇ is used for the operator that differentiates everything that comes after it with respect to u. Making use of Novikov's theorem [51] and van Kampen's lemma [52], we then derive the average equation for ρ [$\langle \rho \rangle = P(u_1, u_2, t)$], the probability density of $u(t)$] as

$$\frac{\partial P(u,t)}{\partial t} = \left\{ A_0 + \alpha^2 \int_0^\infty d\tau \langle A_1(t)\exp(\tau A_0)A_1(t-\tau)\rangle \times \exp(-\tau A_0) \right\} P(u,t). \tag{58}$$

The above result is based on second order cumulant expansion and is valid for the rapid fluctuations with small strength where the correlation time τ is short but finite; i.e.,

$$\langle A_1(t)A_1(t')\rangle = 0 \quad \text{for } |t-t'| > \tau.$$

Equation (58) is exact in the limit $\tau \to 0$. Using the expansion for A_0 and A_1 we obtain

$$\frac{\partial P(u,t)}{\partial t} = -\left\{ \nabla \cdot G_0 + \alpha^2 \int_0^\infty d\tau \langle \nabla \cdot G_1(t) \times \exp(-\tau \nabla \cdot G_0) \nabla \cdot G_1(t-\tau)\rangle \times \exp(\tau \nabla \cdot G_0) \right\} P(u,t). \tag{59}$$

The operator $\exp(-\tau \nabla \cdot G_0)$ in the above equation provides the solution of the equation

$$\frac{\partial \wp(u,t)}{\partial t} = -\nabla \cdot G_0 \wp(u,t). \tag{60}$$

Here $\wp$ signifies the unperturbed part of ρ, which can be found explicitly in terms of characteristic curves. The equation

$$\dot{u} = G_0(u) \tag{61}$$

for fixed t determines a mapping from $u(\tau=0)$ to $u(\tau)$, i.e., $u \to u^\tau$ with the inverse $(u^\tau)^{-\tau} = u$. The solution of Eq. (60) is given by

$$\wp(u,t) = \wp(u^{-t},0) \left| \frac{d(u^{-t})}{d(u)} \right| = \exp(-t \nabla \cdot G_0)\wp(u,0), \tag{62}$$

where $\left|\frac{d(u^{-t})}{d(u)}\right|$ is a Jacobian determinant. The effect of $\exp(-t\nabla \cdot G_0)$ on $\wp(u)$ is given by

$$\exp(-t \nabla \cdot G_0)\wp(u,0) = \wp(u^{-t},0) \left| \frac{d(u^{-t})}{d(u)} \right|. \tag{63}$$

The above simplification when incorporated in Eq. (59) yields

$$\frac{\partial P}{\partial t} = \nabla \cdot \left\{ -G_0 + \alpha^2 \int_0^\infty d\tau \left| \frac{d(u^{-\tau})}{d(u)} \right| \langle G_1(u,t) \nabla_{-\tau} \cdot G_1(u^{-\tau}, t-\tau)\rangle \left| \frac{du}{du^{-\tau}} \right| \right\} P(u,t), \tag{64}$$

where ∇_τ denotes differentiation with respect to $(u_{-\tau})$. We put $\alpha=1$ for the rest of the treatment. We now identify

$$u_1 = q, \quad u_2 = p,$$

$$G_{01} = p, \quad G_{11} = 0,$$

$$G_{02} = -U'(q) - \Gamma h(q) p,$$

$$G_{12} = g(q)\xi(t). \tag{65}$$

In this notation, Eq. (64) reduces to

$$\frac{\partial P(q,p,t)}{\partial t} = -\frac{\partial}{\partial q}(pP) + \frac{\partial}{\partial p}\{U'(q) + \Gamma h(q) p\} P + \frac{\partial}{\partial p}\int_0^\infty d\tau \left\langle [g(q)\xi(t)] \left[\frac{\partial}{\partial p^{-\tau}} \{g(q^{-\tau}) \times \xi(t-\tau)\} \right] \right\rangle P, \tag{66}$$

where we have used the fact that the Jacobian obeys the equation

$$\frac{d}{dt} \ln \left| \frac{d(q^t, p^t)}{d(q,p)} \right| = \frac{\partial p}{\partial q} + \frac{\partial}{\partial p}[-\Gamma h(q) + U'(q)] = -\Gamma h(q)$$

so that the Jacobian becomes $\exp[-\Gamma h(q) t]$. As a next approximation we consider the "unperturbed" part of Eq. (54) and take the variation of p during τ into account to first order in τ. Thus we have

$$q^{-\tau} = q - \tau p,$$

$$p^{-\tau} = p + \Gamma h(q) \tau p + \tau U'(q). \tag{67}$$

Neglecting terms $O(\tau^2)$, Eq. (67) can be simplified after some algebra to the following form:

$$\frac{\partial P(q,p,t)}{\partial t} = -p\frac{\partial p}{\partial q} + [\Gamma h(q) p + U'(q) - 2g(q)g'(q)J_e]\frac{\partial P}{\partial p} + A\frac{\partial^2 P}{\partial p^2} + B\frac{\partial^2 P}{\partial q \partial p} + \Gamma h(q) P, \tag{68}$$

where

$$A(q) = g^2(q) I_e - \Gamma h(q) g^2(q) J_e,$$

$$B = g^2(q) J_e,$$

$$I_e = \int_0^\infty \langle \xi(t)\xi(t-\tau)\rangle d\tau,$$

$$J_e = \int_0^\infty \tau \langle \xi(t)\xi(t-\tau)\rangle d\tau.$$

From Eq. (39) we have $I_e = D_R$ and $J_e = \tau_e D_R$. Thus, Eq. (68) can be written as

$$\frac{\partial P(q,p,t)}{\partial t} = -p\frac{\partial P}{\partial q} + [\Gamma h(q) p + U'(q) - 2g(q)g'(q)\tau_R D_R]\frac{\partial P}{\partial p} + [g^2(q)\tau_R - \Gamma h(q) g^2(q) \tau_R D_r]\frac{\partial^2 P}{\partial p^2} + g^2(q)\tau_R D_R \frac{\partial^2 P}{\partial q \partial p} + \Gamma h(q) P. \tag{69}$$

In our above analysis, $\eta(t)$ and $\epsilon(t)$ are assumed to be uncorrelated as they have different origin. Also, it should be noted that the above Fokker-Planck equation is valid for small but finite correlation time. The fourth term in Eq. (69) is a non-Markovian contribution for finite correlation time of external noise. For small τ_R we may neglect this term to get the approximate Fokker-Planck equation in "phase space" as

$$\begin{aligned}\frac{\partial P(q,p,t)}{\partial t} &= -\frac{\partial(pP)}{\partial q} + \frac{\partial}{\partial p}[\Gamma h(q)p + U'(q) \\ &\quad - 2g(q)g'(q)\tau_R D_R]P \\ &\quad + [g^2(q)\tau_R - \Gamma h(q)g^2(q)\tau_R D_R]\frac{\partial^2 P}{\partial p^2}. \qquad (70)\end{aligned}$$

In terms of an auxiliary function $G(q)$ and a stationary Gaussian δ-correlated fluctuating force $\beta(t)$, the above Fokker-Planck equation, Eq. (70) can be equivalently described by the Langevin equation [53]

$$\dot{q} = p,$$

$$\dot{p} = -U'(q) - \Gamma h(q)p + G(q)\beta(t), \qquad (71)$$

where $G(q)=g(q)\sqrt{[1-\Gamma h(q)\tau_R]}$ and the statistical properties of the auxiliary noise are given by

$$\langle\beta(t)\rangle = 0,$$

$$\langle\beta(t)\beta(t')\rangle = 2D_R\delta(t-t'). \qquad (72)$$

The method of Sancho and co-workers [50] is followed further to obtain the ordinary Stratonovich description [54] of the overdamped Langevin equation in the medium where friction is state dependent as

$$\dot{q} = -\frac{V'(q)-Q_V}{\Gamma h(q)} - D_R\frac{G(q)G'(q)}{\Gamma[h(q)]^2} + \frac{G(q)}{\Gamma h(q)}\beta(t). \qquad (73)$$

The corresponding Fokker-Planck-Smoluchowski equation for the probability density $P(q,t)$ of a particle to be at q at a time t is

$$\begin{aligned}\frac{\partial P(q,t)}{\partial t} &= \frac{\partial}{\partial q}\left[\frac{V'(q)-Q_V}{\Gamma h(q)}\right]P(q,t) \\ &\quad + D_R\frac{\partial}{\partial q}\left[\frac{G(q)G'(q)}{\Gamma h(q)^2}\right]P(q,t) \\ &\quad + D_R\frac{\partial}{\partial q}\left[\frac{G(q)}{\Gamma h(q)}\frac{\partial}{\partial q}\frac{G(q)}{\Gamma h(q)}\right]P(q,t) \qquad (74)\end{aligned}$$

which can be written in a more compact form as

$$\frac{\partial P(q,t)}{\partial t} = \frac{\partial}{\partial q}\frac{1}{\Gamma h(q)}\left[V'(q) - Q_V + \frac{D_R}{\Gamma}\frac{\partial}{\partial q}\frac{G^2(q)}{h(q)}\right]P(q,t). \qquad (75)$$

Equation (75) is the required Smoluchowski equation corresponding to the quantum Langevin equation where the noise is multiplicative and the dissipation is state dependent and where the system-reservoir combination is not thermodynamically closed; rather the reservoir is extremely modulated by an Ornstein-Uhlenbeck noise.

Under stationary condition, $\frac{\partial P}{\partial t}=0$, Eq. (75) reduces to

$$\frac{D_R}{\Gamma}\frac{d}{dq}\left[\frac{G^2(q)}{h(q)}P_{\mathrm{st}}(q)\right] + [V'(q) - Q_V]P_{\mathrm{st}}(q) = 0, \qquad (76)$$

from which we have the stationary probability distribution in the overdamped limit as

$$P_{\mathrm{st}}(q) = \mathcal{N}\frac{h(q)}{G^2(q)}\exp\left[-\int_0^q \frac{V'(q)-Q_V}{R(q)}dq\right] \qquad (77)$$

with $R(q)=D_R[\frac{G^2(q)}{h(q)}]$ and $\mathcal{N}$ is a normalization constant. It can be shown easily that all quantum corrections Q_V, etc., vanish in the classical regime where the quantum fluctuations around their mean value is zero. When the external noise is absent and for linear system-reservoir coupling, the above equation (77) boils down to the conventional Boltzmann distribution in classical limit: $P_{\mathrm{st}}(q)=\mathcal{N}\exp[-V(q)/k_BT]$. The stationary distribution (77) is essentially a generalization of Boltzmann factor for state-dependent diffusion in a quantum open system. The space-dependent friction arises in an inhomogeneous medium and can be described phenomenologically in several ways. The diffusion term for Brownian particle in inhomogeneous medium may assume several forms, microscopic origin of which do not have a common Hamiltonian. Thus, the physics of diffusion in inhomogeneous media is somewhat model dependent [55]. Also, the diverse forms notwithstanding, the generalization of Boltzman factor $\exp[-V(q)/k_BT]$ for state-dependent diffusion in the steady-state assumes a common structure

$$P_{\mathrm{st}}(q) \sim \exp[-\phi(q)]$$

with

$$\phi(q) = \int_0^q \frac{V'(q)-Q_V}{R(q)}dq,$$

$V(q)$ being the potential field. The above steady-state distribution implies that the effective potential $\phi(q)$ is nonlocal in space. The generality in the structure of $\phi(q)$ is such that it may include the spatial variation of temperature, diffusion, or drift coefficient as specific cases as considered separately by several authors [18,21,55]. In the Langevin scheme of description, on the other hand, state-dependent diffusion has received attention under multiplicative noises [23]. The microscopic origin of multiplicative noise within the framework of standard paradigm of system-reservoir Hamiltonian that includes a variety of model calculations is the nonlinear coupling between the system and the bath coordinates which leads to nonlinear dissipation. A thermodynamically consistent approach in this context was put forward by Lindenberg and co-workers [56]. An exact Fokker-Planck equation for time- and space-dependent friction was derived by Pollak *et al.* [57]. Along with these formal developments, the theories of multiplicative noise have found wide applications in several areas, e.g., activated processes [58], stochastic resonance [59], laser and optics [60], signal processing [61], noise-induced transport [62], noise-induced transitions [63], etc. We extend the above studies to a thermodynamically open system in the context of directed transport phenomena in the quantum-mechanical regime. The system is thermodynamically open in the sense that the associated quantum heat bath is modulated by an external colored noise.

IV. PERIODIC POTENTIAL AND PHASE-INDUCED CURRENT

In the overdamped limit, i.e., for large damping, traditionally one eliminates the fast variables $\dot{p}$ adiabatically [49] by simply putting $\dot{p}=0$. This adiabatic elimination provides the correct equilibrium distribution only when dissipation is state independent. But for state-dependent dissipation, we have to resort the approach of Sancho *et al.* [50]. Thus, in this limit, using Eq. (76), the stationary current can be written as

$$J=-\frac{1}{\Gamma h(q)}\left[\frac{D_R}{\Gamma}\frac{d}{dq}\frac{G^2(q)}{h(q)}+[V'(q)-Q_V]\right]P_{\rm st}(q). \quad (78)$$

Integrating the above equation we have the expression of stationary probability density function in terms of stationary current as

$$P_{\rm st}(q)=\frac{e^{-\phi(q)}\times h(q)}{G^2(q)}\left[\frac{G^2(0)}{h(0)}P_{\rm st}(0)-J\frac{\Gamma^2}{D_R}\int_0^q h(q')e^{\phi(q')}dq'\right], \quad (79)$$

where

$$\phi(q)=\frac{\Gamma}{D_R}\int_0^q\frac{V'(q)-Q_V}{[1-\Gamma h(q)\tau_R]g^2(q)}dq$$

is the effective potential. The spatial asymmetry of $\phi(q)$ makes the left-right flux unequal and provides a nonvanishing stationary current. We now consider a symmetric periodic potential with periodicity 2π $V(q+2\pi)=V(q)$ and the periodic derivative of coupling function with the same periodicity $f'(q+2\pi)=f'(q)$. Now, applying the periodic boundary condition on $P_{\rm st}(q)$, $P_{\rm st}(q+2\pi)=P_{\rm st}(q)$, we have from Eqs. (79)

$$\frac{G^2(0)}{h(0)}P_{\rm st}(0)=J\frac{\Gamma^2}{D_R}\left[\frac{1}{1-e^{\phi(2\pi)}}\right]\int_0^{2\pi}h(q)e^{\phi(q)}dq. \quad (80)$$

By applying the normalization condition on stationary probability distribution given by $\int_0^{2\pi}P_{\rm st}(q)dq=1$, we get from Eq. (79)

$$\int_0^{2\pi}\frac{e^{-\phi(q)}\times h(q)}{G^2(q)}\left[\frac{G^2(0)}{h(0)}P_{\rm st}(0)-J\frac{\Gamma^2}{D_R}\int_0^q h(q)e^{\phi(q)}\right]dq=1. \quad (81)$$

Now eliminating $\frac{G^2(0)}{h(0)}P_{\rm st}(0)$ from Eq. (80) and (81) we have the expression for stationary current as

$$J=\frac{D_R}{\Gamma^2}(1-e^{\phi(2\pi)})\times\left\{\int_0^{2\pi}\frac{h(q)}{G^2(q)}e^{-\phi(q)}dq\int_0^{2\pi}h(q)e^{\phi(q)}dq-[1-e^{\phi(2\pi)}]\int_0^{2\pi}\frac{h(q)}{G^2(q)}e^{-\phi(q)}\int_0^q h(q')e^{\phi(q')}dq'dq\right\}^{-1}. \quad (82)$$

From the condition of periodicity of potential and different quantum correction terms it is clear that for the periodic potential and the periodic derivative of coupling function with the same period $\phi(2\pi)=0$ and, consequently, the current reduces to zero. Thus, one can conclude that no current will be generated for a periodic potential and periodic derivative of coupling with same periodicity since there is no symmetry breaking mechanism.

At this point it is important to calculate the quantum correction terms. Following Ray *et al.* [48], the details of the calculations of quantum correction terms are shown in Appendix A. Though the quantum dispersion terms $\langle\delta\hat{q}^n\rangle_Q$ can be obtained by direct numerical simulation of the coupled Eq. (A3) subject to appropriate boundary conditions, it is instructive to deal with quantum correction terms in the analytical way to find out the approximate value of quantum dispersion terms. For the overdamped limit we neglect the $\delta\hat{p}$ term from Eq. (A2) to obtain

$$\frac{d}{dt}\delta\hat{q}=\frac{1}{\Gamma[f'(q)]^2}[-V''(q)\delta\hat{q}-2\Gamma pf'(q)f''(q)\delta\hat{q}+\xi(t)f''(q)\delta\hat{q}]+O(\delta\hat{q}^2). \quad (83)$$

With the help of Eq. (83) we then obtain the equations for $\langle\delta\hat{q}^n\rangle_Q$ in the lowest order

$$\frac{d}{dt}\langle\delta\hat{q}^2\rangle_{\rm Q}=\frac{2}{\Gamma[f'(q)]^2}[-V''(q)\langle\delta\hat{q}^2\rangle_Q-2\Gamma pf'(q)f''(q)\delta\hat{q}+\xi(t)f''(q)\langle\delta\hat{q}^2\rangle_Q], \quad (84)$$

where we have neglected the terms $O(\langle\delta\hat{q}^3\rangle_Q)$. A simplified expression for the leading order quantum correction term $\langle\delta\hat{q}^2\rangle_Q$ can be estimated by neglecting the higher-order coupling terms in the square bracket in Eq. (84) and rewriting it as

$$d\langle\delta\hat{q}^2\rangle_Q=-\frac{2}{\Gamma[f'(q)]^2}V''(q)\langle\delta\hat{q}^2\rangle_Q dt.$$

On the other hand, the overdamped deterministic classical motion gives

$$dq=-\frac{V'(q)}{\Gamma[f'(q)]^2}dt.$$

These together yield after integration

$$\langle\delta\hat{q}^2\rangle_Q=\Delta_q[V'(q)]^2, \quad (85)$$

where

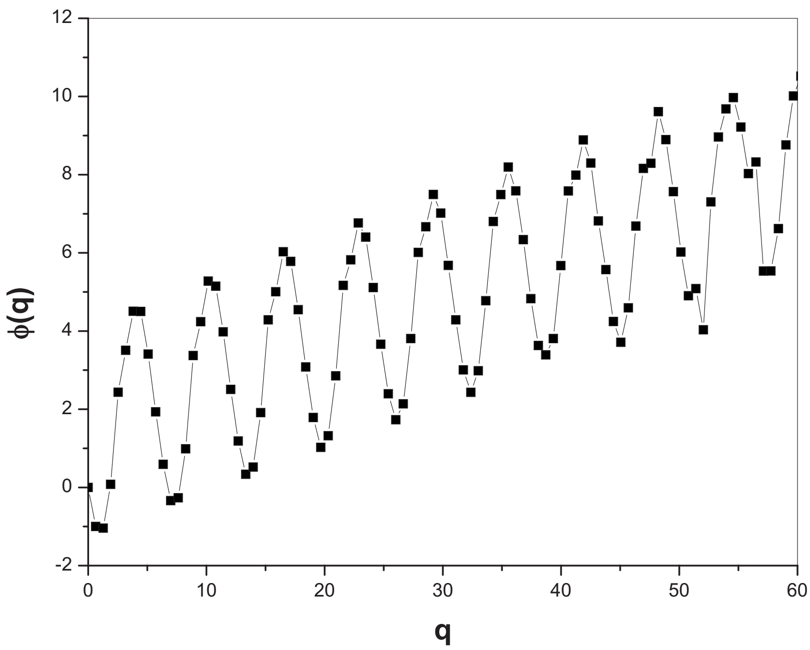

FIG. 1. Plot of effective potential $[\phi(q)]$ with external noise strength $(D)=1.0$, correlation time of external noise $(\tau)=0.2$ and phase difference between coupling function and periodic potential $(\theta)=0.69\pi$.

$$\Delta_q = \frac{\langle \delta\hat{q}^2\rangle_Q^0}{[V'(q)]^2} \tag{86}$$

and q^0 is a quantum-mechanical mean position at which $\langle \delta\hat{q}^2\rangle_Q$ becomes minimum: $\langle \delta\hat{q}^2\rangle_Q^0 = \frac{\hbar}{2\omega_0}$, ω_0 being defined earlier.

For numerical implementation of our result, Eq. (82), we consider a sinusoidal periodic and symmetric potential

$$V(q) = V_0[1+\cos(q+\theta)], \tag{87}$$

where V_0 is the barrier height and θ is the phase factor which can be controlled externally. The coupling function is chosen as $f(q)=(q+\alpha\sin q)$ so that the derivative of the coupling function becomes $f'(q)=(1+\alpha\cos q)$, where α is the modulation parameter. Consequently, from Eq. (85), the second order quantum correction in the over damped limit becomes $\langle \delta\hat{q}^2\rangle_Q = -\Delta_q V_0^2 \sin^2(q+\theta)$, and the correction to the potential in the leading order is given by

$$Q_V = -\frac{1}{2}\Delta_q V_0^3 \sin^3(q+\theta). \tag{88}$$

The quantum correction Q_f and Q_3 in the same order can be estimated as

$$Q_f = -\frac{1}{2}\Delta_q \alpha V_0^2 \cos q \sin^2(q+\theta),$$

$$Q_3 = \Delta_q \alpha^2 V_0^2 \sin^2 q \sin^2(q+\theta). \tag{89}$$

Further, we calculate the functions $h(q)$ and $g(q)$ using Eq. (89) as

$$h(q) = (1+\alpha\cos q)^2 - \Delta_q \alpha V_0^2 \cos q \sin^2(q+\theta)(1+\alpha\cos q) + \Delta_q \alpha^2 V_0^2 \sin^2 q \sin^2(q+\theta),$$

$$g(q) = (1+\alpha\cos q) - \frac{1}{2}\Delta_q \alpha V_0^2 \cos q \sin^2(q+\theta). \tag{90}$$

In the limit of $\hbar = k_B = 1$, we set the parameters $\langle \delta\hat{q}^2\rangle_Q^0 = \frac{1}{2}$, the minimum uncertainty value $\Delta_q = 0.5$, $V_0 = 1.0$, $\omega_0 = 1.0$, $\alpha = 1.0$, $T = 0.5$, $\Gamma = 1.0$, $\kappa_0 = 0.2$, and $D = 1.0$. In Fig. 1, we plot the variation of effective potential $\phi(q)$ for a particular phase difference 0.69π, from which we observe a tilt to the effective potential which makes the transition between left to right and right to left unequal. The denominator for the effective potential $\phi(q)$ remains state dependent and, consequently, $\phi(2\pi)$ becomes, in general, nonzero. The consequence of this is the existence of unidirectional mass motion. From the expression of effective potential $\phi(q)$, it is also apparent that $P_{st}(q)$ may even be peaked at positions which would be quite less likely to be populated in the stationary situation, i.e., when $\Gamma(q)$ is not state dependent and external noises are absent, because in this case, $\phi(q)$ depends on various dynamical and kinematical parameters such as $\Gamma(q)$. The variation of current as a function of phase difference is shown in Fig. 2 for four different values of τ with a fixed $D=1.0$. In Fig. 3, we have shown the variation of J with θ for four different D at a fixed value of τ. It is interesting to observe from Figs. 2 and 3 that the current is a periodic function of the phase difference between modulations of potential and diffusion. The amplitude of current increases with the increase in strength of the external noise D. This is due to the fact that the effective temperature of the bath has been increased from its equilibrium temperature, when the bath is modulated by external noise [26]. This is apparent from the expression of D_R [see Eq. (40)]. Also, one can observe from Figs. 2 and 3 that for $\theta=0$, $n\pi$ with $n=\pm1,\pm2,\ldots$, the current vanishes. In Fig. 4, we have plotted the variation of current J as a function of D for four different values of correlation time τ of the external noise $\epsilon(t)$ where the phase difference is kept at $\theta=0.69\pi$. From Fig. 4, it is clear that for

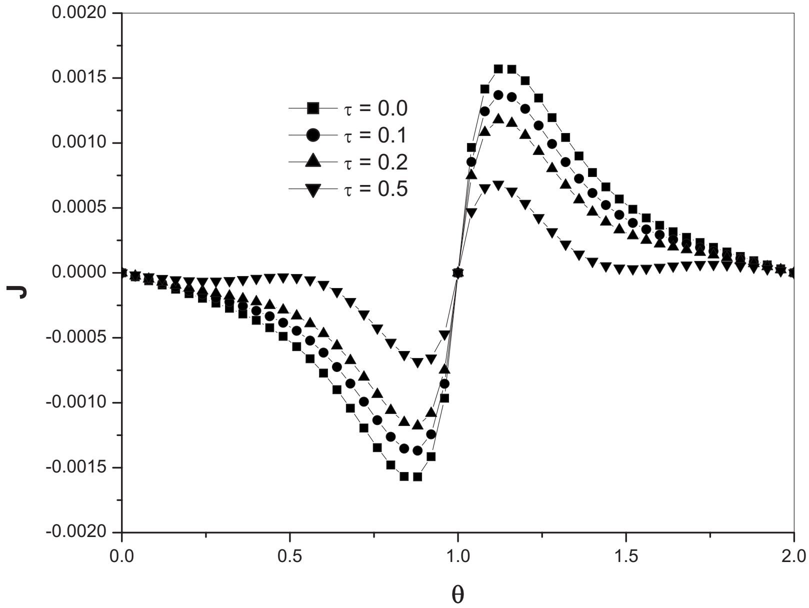

FIG. 2. Variation of current, J with phase difference θ (in unit of π) for various τ, where $D=1.0$.

nonzero τ value, as D increases, the current J increases almost linearly. But for $\tau=0.0$, i.e., for δ-correlated external noise, J is almost independent of D. Also, it is important to observe that when the external noise is absent (i.e., $D=0.0$) there exists a small nonvanishing current due to the quantum fluctuations of the heat bath. Even when $T=0$, this current will exist in the absence of external noise due to the vacuum fluctuation. Figure 5 illustrates the variation of current J as a function of temperature for the phase $\theta=0.69\pi$, $D=1.0$ and for several values of the correlation time τ of the external noise $\epsilon(t)$. From Fig. 5 one observes that even at $T=0$, the vacuum field of the heat bath along with the external fluctuation induces a finite current which reduces with increase in τ value [as is evident from expression of $\phi(q)$] for as τ increases, slope of the effective potential decreases.

From the very mode of development of our formalism and above discussions, it can be extracted that a state-dependent noise and diffusion is generated in a quantum system in the presence of nonlinear system-bath coupling and bath modulation by external colored noise. In a classical system any net directional mass motion [53] will not be created for bath modulation by δ-correlated noise. However, in quantum systems, phase-induced current will be generated due to symmetry breaking of effective potential even when the bath is modulated by white noise $\xi(t)$. This phase-induced current will, however, disappear if the phase bias and modulation of the bath by external noise are absent. We present our observations in Figs. 2 and 3 to substantiate the above discussion. Thus, it is evident that this phase-induced phenomena and the behavior of current on the correlation time of the external noise is exclusively a quantum effect. This is the primary physical significance of our present work present in this paper.

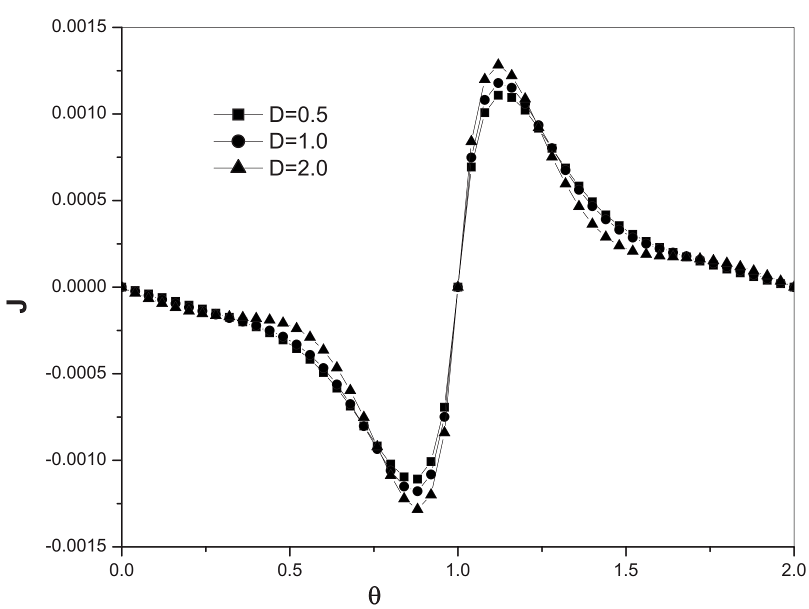

FIG. 3. Plot of current, J as a function of phase difference, θ (in units of π) for different D, where $\tau=0.2$.

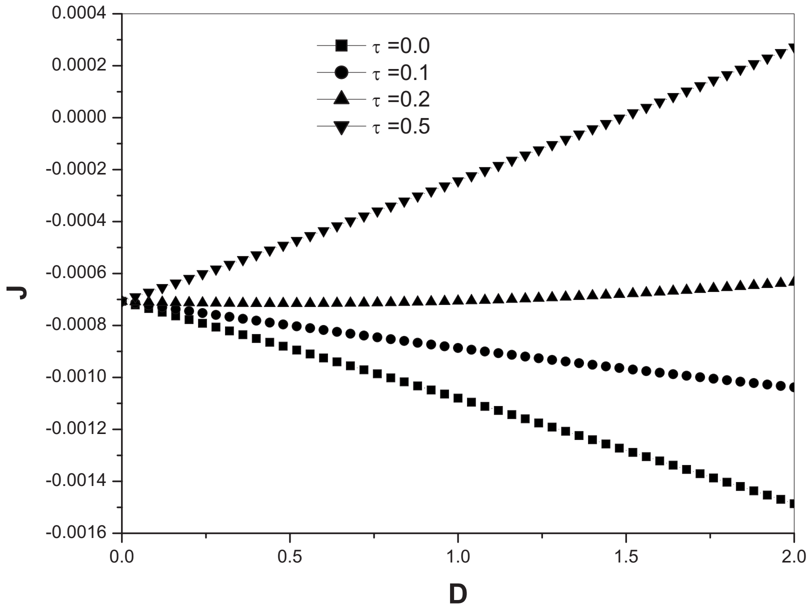

FIG. 4. Plot of current J as a function of D for different τ values with $\theta = 0.69\pi$.

V. CONCLUSION

We have formulated a theory for the diffusion of an open quantum system in inhomogeneous medium. In our formalism, since the associated bath is modulated by an external colored noise with a short but finite correlation time, the quantum system is thermodynamically open. Our approach is based on the system-reservoir model with nonlinear system-bath coupling. We then derive the quantum Langevin equation with multiplicative noises and a nonlinear dissipation. Then, we obtain the C-number analog of the quantum Langevin equation in the Markovian limit. Following Sancho we then derive the quantum analog of the Smoluchowski equation for the state-dependent diffusion of a quantum open system. The openness is due to the modulation of the associated quantum heat bath. It is apparent that the state dependence owes its origin to nonlinear coupling between the system and the bath degrees of freedom. We have applied the formalism to the problem of diffusion of a quantum particle in a periodic potential, where the derivative of coupling function is also periodic with the same periodicity. We observe that a phase difference between these two spatially periodic modulations may give rise to a directed quantum current when the bath is modulated by an externally correlated Ornstein-Uhlenbeck noise. We then numerically examine the behavior of this quantum current for various parameters of external noise. In the classical regime, the net current vanishes if we modulate the bath by δ-correlated noise. However, for a quantum system, modulation by δ-correlated noise breaks down the symmetry of the potential and the generation of quantum current does occur. The effect of the correlation time and the strength of the external noise on the directed motion is also examined and we observe that in the absence of external noise (i.e., $D = 0.0$), there exists a small current

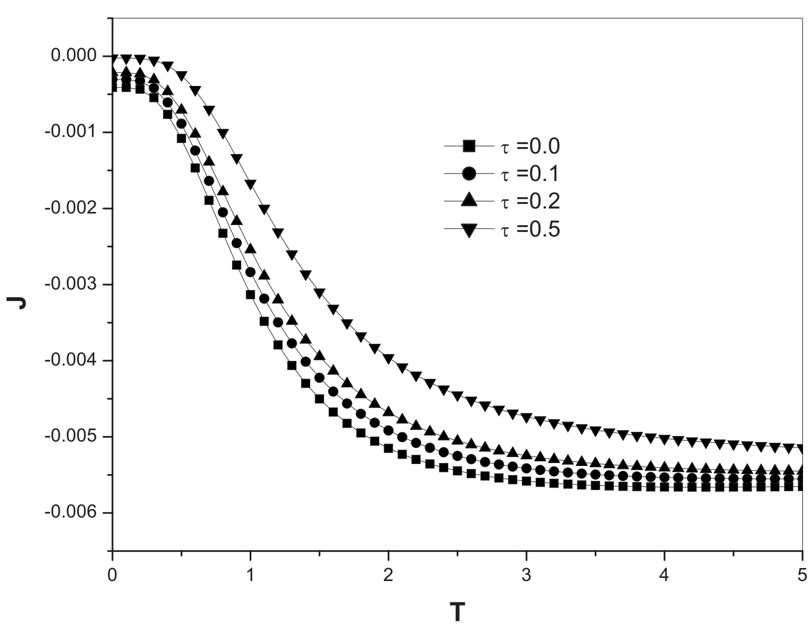

FIG. 5. Plot of current J as a function of temperature T for different τ values with $\theta = 0.69\pi$ and $D = 1.0$.

due to the quantum fluctuations of the heat bath. In contrast to the classical approach, this current will exists in the absence of external noise due to the vacuum fluctuation even when $T=0$. This theory has more room for further development and wide applications. In the presence of the external colored multiplicative noise, we would like to study the barrier crossing dynamics and also examine the quantum current in various ratchet potentials in the near future.

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APPENDIX: QUANTUM CORRECTION TERMS

Here, we address the calculation of the required quantum corrections. The following is a succinct recapitulation of the essential issues pertaining to the same without a detailed derivation. The detailed discussions of the quantum corrections are presented at length in Ref. [44]. In the Heisenberg picture, one can write the system operators $\hat{q}$ and $\hat{p}$ as $\hat{q}=q+\delta\hat{q}$ and $\hat{p}=p+\delta\hat{p}$, respectively. $\delta\hat{q}$ and $\delta\hat{p}$ describe the quantum fluctuations around their respective mean values.

With the help of the operator Langevin equations (8) in the Markovian limit, the time evolution of these correction terms can be calculated via the following equations using quantum-mechanical average over the initial product separable coherent bath states:

$$\dot{\hat{q}} = \hat{p},$$

$$\dot{\hat{p}} = -V'(\hat{q}) - \Gamma[f'(\hat{q})]^2\hat{p} + f'(\hat{q})\eta(t) + f'(\hat{q})\pi(t), \quad \text{(A1)}$$

$$\delta\dot{\hat{q}} = \delta\hat{p},$$

$$\begin{aligned}\delta\dot{\hat{p}} = &-V''(q)\delta\hat{q} - \sum_{n\geq 2}\frac{1}{n!}V^{n+1}(q)[\delta\hat{q}^n - \langle\delta\hat{q}^n\rangle_Q] \\ &- \gamma\Bigg[2f'(q)f''(q)\delta\hat{q} + 2f'(q)\sum_{n\geq 2}\frac{1}{n!}f^{n+1}(q)[\delta\hat{q}^n - \langle\delta\hat{q}^n\rangle_Q] \\ &+ \sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m!}\frac{1}{n!}f^{m+1}(q)f^{n+1}(q)[\delta\hat{q}^m\delta\hat{q}^n - \langle\delta\hat{q}^m\delta\hat{q}^n\rangle_Q]\Bigg]p \\ &- \gamma\Bigg[[f'(q)]^2\delta\hat{p} + 2f'(q)\sum_{n\geq 1}\frac{1}{n!}f^{n+1}(q)[\delta\hat{q}^n\delta\hat{p} \\ &- \langle\delta\hat{q}^n\delta\hat{p}\rangle_Q] + \sum_{m\geq 1}\sum_{n\geq 1}\frac{1}{m!}\frac{1}{n!}f^{m+1}(q)f^{n+1}(q)[\delta\hat{q}^m\delta\hat{q}^n\delta\hat{p} \\ &- \langle\delta\hat{q}^m\delta\hat{q}^n\delta\hat{p}\rangle_Q]\Bigg] + \eta(t)\Bigg[f''(q)\delta\hat{q} + \sum_{n\geq 2}\frac{1}{n!}f^{n+1}(q) \\ &\times[\delta\hat{q}^n - \langle\delta\hat{q}^n\rangle_Q]\Bigg]. \end{aligned} \quad \text{(A2)}$$

From the work of Ray and co-workers [44], it is clear that the operator correction equations can be used to yield an infinite hierarchy of equations. Up to third order, we construct, for example, the following set of equations which are coupled to quantum Langevin equations. From Eq. (8),

$$\frac{d}{dt}\langle\delta\hat{q}^2\rangle_Q = \langle\delta\hat{q}\delta\hat{p} + \delta\hat{p}\delta\hat{q}\rangle_Q,$$

$$\begin{aligned}\frac{d}{dt}\langle\delta\hat{q}\delta\hat{p} + \delta\hat{p}\delta\hat{q}\rangle_Q = &-2\chi(q,p)\langle\delta\hat{q}^2\rangle_Q + 2\langle\delta\hat{q}^2\rangle_Q \\ &- \gamma[f'(q)]^2\langle\delta\hat{q}\delta\hat{p} + \delta\hat{p}\delta\hat{q}\rangle_Q - \zeta(q,p) \\ &\times\langle\delta\hat{q}^3\rangle_Q - 2\gamma f'(q)f''(q)\langle\delta\hat{q}^2\delta\hat{p} \\ &+ \delta\hat{p}\delta\hat{q}^2\rangle_Q,\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\delta\hat{p}^2\rangle_Q = &-2\gamma[f'(q)]^2\langle\delta\hat{p}^2\rangle_Q - \chi(q,p)\langle\delta\hat{q}\delta\hat{p} + \delta\hat{p}\delta\hat{q}\rangle_Q \\ &- \frac{1}{2}\zeta(q,p)\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\delta\hat{q}^2\rangle_Q - 2\gamma f'(q)f''(q) \\ &\times\langle\delta\hat{q}\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q,\end{aligned}$$

$$\frac{d}{dt}\langle\delta\hat{q}^3\rangle_Q = \frac{3}{2}\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\delta\hat{q}^2\rangle_Q,$$

$$\frac{d}{dt}\langle\delta\hat{p}^3\rangle_Q = -3\gamma[f'(q)]^2\langle\delta\hat{p}^3\rangle_Q - \frac{3}{2}\chi(q,p)\langle\delta\hat{q}\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q,$$

$$\begin{aligned}\frac{d}{dt}\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\delta\hat{q}^2\rangle_Q = &-2\chi(q,p)\langle\delta\hat{q}^3\rangle_Q + 2\langle\delta\hat{q}\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q \\ &- \gamma[f'(q)]^2\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\delta\hat{q}^2\rangle_Q,\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\langle\delta\hat{q}\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q = &\,2\langle\delta\hat{p}^3\rangle_Q - 4\chi(q,p)\langle\delta\hat{q}^2\delta\hat{p} + \delta\hat{p}\delta\hat{q}^2\rangle_Q \\ &- 2\gamma[f'(q)]^2\langle\delta\hat{q}\delta\hat{p}^2 + \delta\hat{p}^2\delta\hat{q}\rangle_Q, \end{aligned} \quad \text{(A3)}$$

where

$$\chi(q,p) = V''(q) + 2\gamma p f'(q)f''(q) - \eta(t)f''(q),$$

$$\zeta(q,p) = V'''(q) + 2\gamma p f'(q)f'''(q) + 2\gamma p[f''(q)]^2 - \eta(t)f'''(q).$$

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